of $\delta$ was used to initialize computations. Thus, further acceleration of the technique could easily be achieved taking better initial values of $F_G$ and $V_G$. Table 1 shows that the time (or number of iterations) rises only slightly for initial $\delta$ guesses as much as $\pm$ 30% in error. Below a certain $\delta$ value, the process converges to the positive-flow case rapidly (as shown by Fig 1). In some cases, a guess of 50% to 100% high on $\delta$ was found to still converge to the reversed-flow profile. Thus, compared to the usual shooting technique (where an initial boundary condition is guessed), this process is relatively insensitive to initialization and can be made to converge quickly to either solution branch.

**Table 1 Input error influence on iteration scheme**

| | Time for convergence, sec | | | | | |
|---|---|---|---|---|---|---|
| $\beta$ | $\delta$ input error | | | | | |
| | +10% | −10% | +20% | −20% | +30% | −30% |
| −0.04 | 15 | 15 | 16 | 17 | 18 | 27 |
| −0.08 | 14 | 13 | 18 | 17 | 14 | 23 |

As a final note, it is pointed out that solutions were also obtained to an uncoupled (through $V$) version of Eqs. (8) and (16). This approach as used by Blottner[1] and others employs a quasi-linear version of the momentum equation that only contains $V$ in its guessed state, $V_G$. This method was also successful for present purposes but it was found that the computer time to convergence was prohibitively large for the reversed-flow cases where $-0.12 \leqq \beta < 0$.

It is believed that the present technique provides a rapid, effective means of solving nonunique boundary-layer problems as applied here to the Falkner-Skan equations. As such it provides a basis for addressing nonsimilar separating boundary-layers where it is anticipated approximations similar to those of Reyhner and Flügge-Lotz[1] will be required to overcome the physical reverse flow instability.

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# Approximate Temperature Distribution for a Diffuse, Highly Reflecting Material


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### Nomenclature

$B$ = Biot modulus, $h\delta/K$
$h$ = convective heat-transfer coefficient
$I_R$ = reflecting flux
$I_T$ = transmitted flux
$k$ = Kubelka-Munk absorption coefficient
$K$ = thermal conductivity
$q_o$ = incident radiative heat flux
$q_R$ = radiative heat flux
$R$ = reflectance of material
$R_B$ = reflectance of back surface
$R_\infty$ = reflectance of a slab of infinite thickness
$r$ = $I_R/I_T$
$s$ = Kubelka-Munk scattering coefficient
$S$ = scattering power, $s\delta$
$T$ = temperature
$T_s$ = stream temperature
$u$ = internal energy
$y$ = coordinate normal to surface of material
$\delta$ = slab thickness
$\eta$ = dimensionless distance, $y/\delta$
$\theta$ = nondimensional temperature $(T-T_o)/T_s$
$\rho$ = density
$\Phi$ = nondimensional parameter, $kq_o\delta^2/KT_s$

### Introduction

IT has been proposed[1] that highly-backscattering materials may provide efficient thermal protection for probes encountering planetary entry environments in which radiative heating is the dominant mode of heat transfer. Radiative transfer in such materials may be studied by use of the equation of radiative transfer as discussed in Chandrasekhar[2] and elsewhere. For engineering purposes however the "two-flux method" is convenient and informative. This approach to radiative transfer has been used by astrophysicists[3] in the paint and paper industry[4,5] and in the analysis of thermal control coatings.[6,7] The present work considers applications of the "two-flux" technique to the reflective characteristics and temperature distribution in highly-scattering materials with vanishingly small absorption coefficient. It is assumed here that internal radiative emission may be neglected with respect to the transmitted internal radiation fields.

### Two-Flux Radiation Field

The radiation field is assumed to be governed by the Kubelka-Munk differential equations.[4] These equations describe the diminution of right and leftgoing fluxes $I_T$ and $I_R$ due to scattering


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and absorption and their augmentation due to scattering from the opposing flux

$$dI_T/dy = -(s + k)\, I_T + sI_R \tag{1a}$$

$$dI_R/dy = (s + k)\, I_R - sI_T \tag{1b}$$

where $s$ and $k$ are the Kubelka-Munk scattering and absorption coefficients, respectively.

Following Steele[4] we may define a variable $r = I_R/I_T$ which allows combination into a single differential equation:

$$dr/dy = -s[r^2 - 2(1 + k/s)r + 1] \tag{2}$$

If the surface reflectivity may be neglected, the reflectance of a thickness $\delta$ of scattering and absorbing material may be obtained by integrating Eq. (2) using the following conditions

$$y = 0 \quad r = R = \text{ reflectance of material}$$

$$y = \delta \quad r = R_B = \text{back surface reflectance}$$

to obtain

$$R = \frac{(1/R_\infty)(R_B - R_\infty) - R_\infty(R_B - 1/R_\infty)\exp[s\delta(1/R_\infty - R_\infty)]}{(R_B - R_\infty) - (R_B - 1/R_\infty)\exp[s\delta\,(1/R_\infty - R_\infty)]} \tag{3}$$

where

$$R_\infty = 1 + k/s - [k/s(k/s + 2)]^{1/2} \tag{4}$$

is the reflectance of a slab of infinite thickness. Note that $R_\infty$ is independent of the back surface reflectance and depends only on the ratio of the absorption to the scattering coefficient. It is clear from Eq. (4) that as the ratio $k/s$ approaches zero the reflectance of the infinite slab approaches unity. Thus good diffuse reflectors have very low radiation absorption and densely distributed scattering centers.

It is clear from Eq. (3) that the reflectance of a finite slab depends on the back surface reflectance, the infinite slab reflectance and the product of the scattering coefficient and the slab thickness $s\delta$. The product $s\delta$ is known as the scattering power $S$ of the material. As the scattering power increases for a given absorption coefficient, the reflectance of the finite slab approaches $R_\infty$.

### Comparison with Experiment

In order to demonstrate the adequacy of the Kubelka-Munk relations to represent the reflectance of a diffusely reflecting material, measurements for the reflectance of aluminium oxide over a copper substrate[8] were considered. These data are compared with points computed from Eq. (3) for $s = 17\ \text{mm}^{-1}$, $k = 0.059\ \text{mm}^{-1}$ and $R_B = .38$ in Fig. 1. These values were obtained by trial and error fitting of the data using Eq. (3). The reasonable agreement verifies the validity of the use of the Kubelka-Munk relations in representing the reflectance of diffuse reflectors. For this case the ratio $k/s = 0.00347$ is typical of the low values needed to attein high reflectivity.

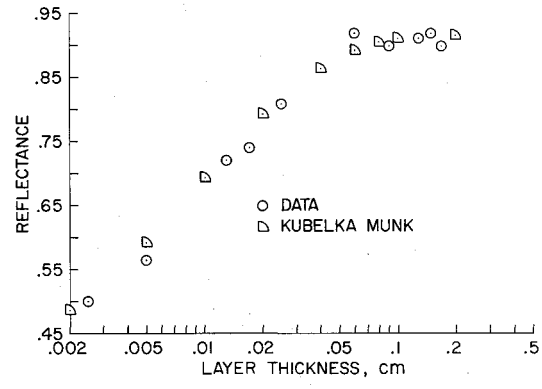


**Fig. 1 Reflectance of aluminium oxide over a copper substrate, comparison of theory and experiment.**

### Reflectance for Nonabsorbing Materials

The nonabsorbing limit for the reflectance of a diffuse reflector is particularly relevant to this discussion. Equation (2) may be integrated for $k = 0$ in the same manner as discussed earlier to yield the following relation:

$$R = [1 - (1 - R_B)(1 - S)]/[1 + (1 - R_B)\, S] \tag{5}$$

It is evident that $S = 0$ here implies a transparent material while $S \to \infty$ implies a perfect diffuse reflector regardless of the value of the back surface reflectance.

### Approximate Steady-State Temperature Distributions in Diffuse Reflectors of Small Absorption Coefficient

The radiation field in a semitransparent continuum may be linked with the temperature field through the divergence of a radiative flux vector in the equation of conservation of energy. In one dimension the energy equation is:

$$\rho(Du/Dt) = [K(d^2T/dy^2) - (dq_R/dy)] \tag{6}$$

where $Du/Dt$, the material derivative of the internal energy, vanishes in the steady state. Defining the radiative flux $q_R$ as $I_T - I_R$ the Kubelka-Munk equations imply that

$$dq_R/dy = dI_T/dy - dI_R/dy = -k(I_T + I_R) \tag{7}$$

It is seen that for a nonabsorbing material $dq_R/dy$ vanishes and there is no interaction between the temperature and radiation fields. We concern ourselves here with the case of small absorption coefficient ($k \ll s$). For such a situation the Kubelka-Munk equations become approximately

$$dI_T/dy = dI_R/dy = -s(I_T - I_R) \tag{8a,b}$$

Taking the boundary conditions as

$$y = 0 \quad I_T = q_o = \text{incident diffuse flux}$$

$$y = 0 \quad I_R(o)/I_T(o) = R = \text{material reflectance}$$

we find

$$I_T = q_o[1 - sy(1 - R)] \tag{9a}$$

$$I_R = q_o[R - sy(1 - R)] \tag{9b}$$

and Eq. (7) becomes:

$$dq_R/dy = -kq_o[1 + R - 2sy(1 - R)] \tag{10}$$

Using (10) the steady-state solution of (6) is

$$T = [skq_o(1 - R)/3K]\, y^3 - [kq_o(1 + R)/2K]\, y^2 + K_1 y + K_2 \tag{11}$$

We consider here the case of specified front surface temperature, specified adjacent stream temperature, and specified heat-transfer coefficient. Thus our boundary conditions are

$$y = 0 \qquad T = T_o$$

$$y = 0 \qquad h(T_s - T_o) = -K(dT/dy)|_{y=0}$$

Defining the parameters:

$$\Phi = kq_o\delta^2/KT_s, \quad B = h\delta/K$$

and the variables $\eta = y/d$ and $\theta = (T - T_o)/T_s$ Eq. (11) becomes:

$$\theta = -B\theta_s\eta - \frac{\Phi}{2}\left[\frac{1 + R_B + 2S(1 - R_B)}{1 - (R_B - 1)\, S}\right]\eta^2 + \frac{\Phi}{3}\left[\frac{S(1 - R_B)}{1 - (R_B - 1)S}\right]\eta^3 \tag{12}$$

where we have used the expression (5) for the material reflectance.

It is clear that, when the incident radiative heat flux vanishes or the absorption coefficient is zero, Eq. (12) reduces to the steady-state temperature distribution for pure conduction. Thus the second and third terms on the right represent the contribution of the radiation field to the temperature distribution.

It should be recognized that this temperature distribution is approximate in that it may be expected to be valid only for small values of the absorption coefficient. The parameter $\Phi$ however may be large when representing large thicknesses and/or low thermal conductivity.

Several nondimensional temperature distributions are shown in Fig. 2 for selected values of the parameters $B\theta_s$, $\Phi$, $R_B$, and $S$. Two families of solutions are evident. For low $R_B$, the temperature

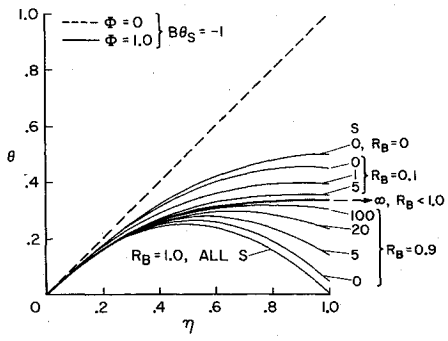


**Fig. 2 Steady-state temperature distribution of a weakly-absorbing diffuse reflector—specified heat-transfer coefficient, stream temperature and interface temperature.**

decreases with increasing $S$ while for high $R_B$ the temperature increases with increasing $S$. In both cases as $S$ increases the temperature gradient at $\eta = 1$ decreases. For all distributions shown the conductive heat fluxes are identical at $\eta = 0$ because of the specification $B\theta_s = -1$. Thus, it is evident that as $S$ increases, the steady-state distribution for each family is characterized by decreased conductive energy loss; therefore the radiation absorbed is reduced. This may be described as a consequence of increasing volume reflectance with increasing scattering coefficient.

A case of particular interest is that of zero conduction to the back surface. We may solve (12) for the value of $B\theta_s$, which satisfies this condition. The resulting temperature distribution is given below and shown in Fig. 3

$$\theta = \Phi\frac{1 + R_B + 2S(1 - R_B)}{1 - (R_B - 1)S}\left\{\left[\frac{1 + R_B + S(1 - R_B)}{1 + R_B + 2S(1 - R_B)}\right]\eta - \frac{1}{2}\eta^2 + \frac{1}{3}\left[\frac{S(1 - R_B)}{1 + R_B + 2S(1 - R_B)}\right]\eta^3\right\} \quad (13)$$

It is evident that in this case the energy absorbed by the diffuser from the radiation field must be conducted out the front surface for energy conservation. Hence the temperature distributions for this case are all monotonic.

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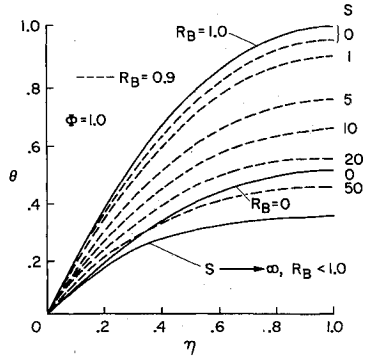


**Fig. 3 Steady-state temperature distribution of a weakly-absorbing diffuse reflector—specified interface temperature and no back surface conduction.**

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# Influence of a Temperature Dependent Spectral Absorption Coefficient on Radiative Flux

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### Introduction

THIS Note reports the results of an investigation of the influence of a temperature dependent absorption coefficient on the continuum radiative flux from a nonisothermal atomic hydrogen shock layer. In a previous paper,[1] the nongray absorption coefficient was assumed to be independent of location in the shock layer and was evaluated at the temperature just behind the shock. Actually, experimental and theoretical work[2] reveals that the absorption coefficient of an atomic gas is a strong function of temperature and, hence, location in the shock layer. In some radiative transfer studies, this presents no difficulty because the shock layer plasma is assumed to be isothermal. Many nonisothermal investigations have included a temperature dependent absorption coefficient, but the necessity of including the temperature dependence in radiative flux calculations has not been investigated.

The analysis is developed in Ref. 1 and is based on the following assumptions: 1) local thermodynamic and chemical equilibrium, 2) one-dimensional, radiative energy transport, 3) negligible radiation emitted from the body, and 4) negligible precursor effects. Line radiation (bound-bound) and the influence of stimulated emission are not considered.

The actual temperature profile in the shock layer can be determined from the coupled conservation equations. Because the objective of the present study is to investigate nongray-nonisothermal radiation, the fluid dynamics and radiation are uncoupled by assuming a linear temperature profile in the shock layer, $T/T_r = \theta(\bar{x}) = \theta_o + (1 - \theta_o)\bar{x}$. This temperature distribution represents the actual temperature profile more accurately than the isothermal approximation, which has been used in many other investigations. The shock wave is located at $\bar{x} = 1$ and the body at $\bar{x} = 0$. The reference temperature $T_r$ is a function of the ambient density and Mach number and is given by the Rankine-Hugoniot equations, and $\theta_o$ is the nondimensional temperature at $\bar{x} = 0$, which is determined by the body's ablating properties.

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